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Measurable cardinals and finite intervals between regular topologies

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Abstract

Assuming the existence of infinitely many measurable cardinals, a finite lattice is isomorphic to the interval between two T_3 topologies on some set if and only if it is distributive. A characterisation is given for those finite lattices which are isomorphic to the interval between two T_3 topologies on a countable set.

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1. Introduction

The set of all topologies on a set X, when ordered by inclusion, forms a complete, bounded lattice, the join of two topologies, $\sigma \vee \tau$, being the topology generated by their union and the meet $\sigma \wedge \tau$ being their intersection. Such lattices were first studied by Birkhoff [1] and have been examined in some detail since (see, for example, the extensive bibliography to Larson and Andima's paper [4]). Here we are interested in finite subintervals of the lattice of topologies. Valent and Larson [6] proved that any finite distributive lattice can be realized as an interval of T_1 topologies, and Rosický [5] proved that any finite interval between T_1 topologies must be distributive. Hence a finite lattice can be realized as such an interval if and only if it is distributive. We ask whether there is a characterization of those finite lattices that are isomorphic to intervals between Hausdorff topologies, or to intervals between T_3 topologies.

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Of course, Rosický's result implies that every finite interval between Hausdorff topologies must be distributive. Since proving Lemma 2 of this paper, the second author, together with Robin Knight and Paul Gartside, has shown that the converse is true [3]. This leaves the question of what happens between T_3 topologies. We will show that all finite distributive lattices occur as intervals between T_3 topologies, assuming the existence of infinitely many measurable cardinals. This may seem a very strong hypothesis: in Section 5 we shall give some indication of why measurable cardinals seem necessary, or at least relevant to the problem. Finally, in Section 7 we discuss the effect of restricting the problem to countable sets.

Note that, in considering intervals between T_i topologies, there is a significant difference between T_2 and T_3: if σ is T_2 then so are all the topologies in $[\sigma, \tau]$. On the other hand, in all but trivial cases, even if σ and τ are both T_3, some of the topologies in $[\sigma, \tau]$ are not T_3: see Section 6.

Our approach is to give a general construction which, given a finite distributive lattice L, will yield topologies σ and τ such that the interval between σ and τ is isomorphic to L. We will then discuss circumstances (essentially the existence of certain ultrafilters) under which the construction can be realized.

2. The basic construction

For a partially ordered set P, let $\mathcal{O}(P)$ denote the set of down-closed subsets of P, partially ordered by inclusion. For L a finite lattice let $J(L)$ denote the set of join-irreducible elements (i.e., elements a such that a is not the least element of the lattice and, if $a = b \vee c$ then $a = b$ or $a = c$). Recall that a finite lattice L is distributive if and only if $L \cong \mathcal{O}(P)$ for some P, which happens if and only if $L \cong \mathcal{O}(J(L))$.

If X is a topological space, then an *o-filter* on X is a filter in the partial order of non-empty open sets of X, and an *o-ultrafilter* is a maximal o-filter. We shall mostly be interested in o-filters on subspaces of X. Since we are considering several topologies on X, there is room for ambiguity here: however, on the subsets we consider the subspace topologies induced by different elements of the lattice will, in fact, coincide, so the ambiguity does not arise. Indeed, we will often be interested in relatively discrete subspaces, in which case the notions of filter and o-filter coincide.

For μ a topology on X and $p \in X$, let $\mathcal{N}_\mu(p)$ denote the neighbourhood filter at p in the topology μ, and let $\mathcal{N}^o_\mu(p)$ denote the o-filter of open neighbourhoods of p in the topology μ (in other words, $\mathcal{N}^o_\mu(p) = \mathcal{N}_\mu(p) \cap \mu$). For $\mathcal{F}$ a filter on X and $A \subseteq X$, let $\mathcal{F} \restriction A$ be the trace of $\mathcal{F}$ on A, in other words the family $\{F \cap A \mid F \in \mathcal{F}\}$. Notice that $\mathcal{N}_\mu(p) \restriction A$ is a proper filter (i.e., does not contain $\emptyset$) if and only if $p \in \overline{A}^\mu$, and similarly for $\mathcal{N}^o_\mu(p) \restriction A$.

If σ is a topology on X and $A \subseteq X$, let $\langle \sigma, A\rangle$ denote the topology which has $\sigma \cup \{A\}$ as a subbasis.

Lemma 1. *Let $\mathcal{F}$ and $\mathcal{G}$ be filters on a set X, and suppose we partition X as $\bigcup_{i<n} X_i$, where $n \in \omega$. If $\mathcal{F} \restriction X_i = \mathcal{G} \restriction X_i$ for each $i < n$ then $\mathcal{F} = \mathcal{G}$.*

Proof. Trivial.

Lemma 2. *Let P be a finite partially ordered set. Suppose we can find a set X of the form $X = \{p\} \cup \bigcup_{a \in P} S_a$, where the sets S_a are disjoint, non-empty and do not contain p, and a topology σ on X such that*

(1) *For each $a \in P$, $\overline{S_a}^{\sigma} = \{p\} \cup \bigcup_{b \leqslant a} S_b$.*

(2) *For each $a \in P$, $\mathcal{N}^o_\sigma(p) \restriction S_a$ is an o-ultrafilter on S_a.*

Let $\tau = \langle \sigma, \{p\} \rangle$. Then $[\sigma, \tau] \cong \mathcal{O}(P)$.

Proof. Define $\Phi : [\sigma, \tau] \to \mathcal{O}(P)$ by

$$\Phi(\mu) = \left\{ a \in P \mid p \notin \overline{S_a}^{\mu} \right\}.$$

Now, Φ is well-defined (i.e., $\Phi(\mu)$ is down-closed for each $\mu \in [\sigma, \tau]$) because if $b \leqslant a$ and $\mu \in [\sigma, \tau]$ then $\overline{S_b}^{\mu} \subseteq \overline{S_a}^{\mu}$.

The fact that Φ is order-preserving follows from the observation that if $\mu, \nu \in [\sigma, \tau]$ with $\mu \leqslant \nu$ then $\overline{S_a}^{\nu} \subseteq \overline{S_a}^{\mu}$.

To show that Φ is 1–1, let $\mu, \nu \in [\sigma, \tau]$ with $\Phi(\mu) = \Phi(\nu)$. We must show that $\mu = \nu$, for which it is sufficient to show that $\mathcal{N}_\mu(x) = \mathcal{N}_\nu(x)$ for every $x \in X$. Now $\mathcal{N}_\mu(x) = \mathcal{N}_\nu(x) = \mathcal{N}_\sigma(x)$ for every $x \in X \setminus \{p\}$, so we only need to consider the neighbourhoods of p. Now, for every $a \in P$, $\mathcal{N}^o_\sigma(p) \restriction S_a \subseteq \mathcal{N}^o_\mu(p) \restriction S_a \subseteq \mathcal{N}^o_\tau(p) \restriction S_a$. Since $\mathcal{N}^o_\sigma(p) \restriction S_a$ is an o-ultrafilter on S_a, we must have either $\mathcal{N}^o_\sigma(p) \restriction S_a = \mathcal{N}^o_\mu(p) \restriction S_a$, or $\mathcal{N}^o_\mu(p) \restriction S_a = \mu \restriction S_a = \sigma \restriction S_a$. The latter holds if and only if $p \notin \overline{S_a}^{\mu}$, i.e., if and only if $a \in \Phi(\mu)$. Similarly for ν. So we have

$$\mathcal{N}^o_\mu(p) \restriction S_a = \begin{cases} \sigma \restriction S_a & \text{if } a \in \Phi(\mu), \\ \mathcal{N}^o_\sigma(p) \restriction S_a & \text{otherwise,} \end{cases}$$
$$\mathcal{N}^o_\nu(p) \restriction S_a = \begin{cases} \sigma \restriction S_a & \text{if } a \in \Phi(\nu), \\ \mathcal{N}^o_\sigma(p) \restriction S_a & \text{otherwise.} \end{cases}$$

Since $a \in \Phi(\mu)$ if and only if $a \in \Phi(\nu)$, we have $\mathcal{N}^o_\mu(p) \restriction S_a = \mathcal{N}^o_\nu(p) \restriction S_a$ for every a, so by Lemma 1 $\mathcal{N}^o_\mu(p) = \mathcal{N}^o_\nu(p)$. Hence $N_\mu(p) = \mathcal{N}_\nu(p)$, as required.

Finally, we must show that Φ is onto. So let $A \subseteq P$ be down-closed. Then

$$\bigcup_{a \in A} S_a = \bigcup_{a \in A} \bigcup_{b \leqslant a} S_b = \bigcup_{a \in A} \overline{S_a}^{\tau}.$$

Put $U = X \setminus \bigcup_{a \in A} S_a$. Then, by the above, $U \in \tau$, so $\mu = \langle \sigma, U \rangle \in [\sigma, \tau]$. We must show that $\Phi(\mu) = A$. Now, if $b \notin A$ then $S_b \subseteq U$, and any neighbourhood of p in μ is either of the form V or $V \cap U$, where $V \in \sigma$. Either way, since any such V must meet S_b, $p \in \overline{S_b}^{\mu}$, so $b \notin \Phi(\mu)$. On the other hand, if $b \in A$ then U is a neighbourhood of p which misses S_b, so $b \in \Phi(\mu)$. Thus $\Phi(\mu) = A$, as required.

3. Products of ultrafilters

In this section we review some ways of constructing ultrafilters on Cartesian products of sets from ultrafilters on the factors. Some of these results appear in [2, pp. 157–161]: their notation is $\bar{t}(\mathcal{V}\cdot\mathcal{U})$ for what we will call $\mathcal{U}*\mathcal{V}$ and $\mathcal{U}\times\mathcal{V}$ for what we will call $\mathcal{U}\otimes\mathcal{V}$.

Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on sets A and B, respectively. Then there are three filters one could naturally define on $A\times B$:

$$\begin{aligned}\mathcal{U}\cdot\mathcal{V} &= \bigl\{S\subseteq A\times B \mid \bigl\{a\in A \mid S^a\in\mathcal{V}\bigr\}\in\mathcal{U}\bigr\},\\ \mathcal{U}*\mathcal{V} &= \bigl\{S\subseteq A\times B \mid \{b\in B \mid S_b\in\mathcal{U}\}\in\mathcal{V}\bigr\},\\ \mathcal{U}\otimes\mathcal{V} &= \bigl\{S\subseteq A\times B \mid (\exists U\in\mathcal{U})\ (\exists V\in\mathcal{V})\ U\times V\subseteq S\bigr\},\end{aligned}$$

where $S^a=\{b\in B \mid \langle a,b\rangle\in S\}$ and $S_b=\{a\in A \mid \langle a,b\rangle\in S\}$. Clearly $\mathcal{U}\otimes\mathcal{V}\subseteq\mathcal{U}\cdot\mathcal{V}$ and $\mathcal{U}\otimes\mathcal{V}\subseteq\mathcal{U}*\mathcal{V}$. It is easy enough to see that $\mathcal{U}\cdot\mathcal{V}$ and $\mathcal{U}*\mathcal{V}$ are both ultrafilters, but that typically $\mathcal{U}\otimes\mathcal{V}$ is not.

It is also easy to see that if $\mathcal{U}$ is κ-complete and $\mathcal{V}$ is λ-complete then $\mathcal{U}\otimes\mathcal{V}$ is $\min(\kappa,\lambda)$-complete, and that if either $\mathcal{U}$ or $\mathcal{V}$ is free then $\mathcal{U}\otimes\mathcal{V}$ is free.

Lemma 3. *If $\mathcal{V}$ is $|A|^+$-complete then $\mathcal{U}\cdot\mathcal{V}=\mathcal{U}\otimes\mathcal{V}$, and hence all three filters are the same ultrafilter.*

Proof. It is enough to show that $\mathcal{U}\cdot\mathcal{V}\subseteq\mathcal{U}\otimes\mathcal{V}$. So let $S\in\mathcal{U}\cdot\mathcal{V}$. Let $U=\{a\in A \mid S^a\in\mathcal{V}\}$. Then $U\in\mathcal{U}$. Put $V=\bigcap_{a\in U}S^a$. Then V is the intersection of fewer than $|A|^+$ many sets in $\mathcal{V}$, so $V\in\mathcal{V}$. Also $U\times V\subseteq S$. Thus $S\in\mathcal{U}\otimes\mathcal{V}$, as required.

To see that all three filters are equal, note that since $\mathcal{U}\otimes\mathcal{V}=\mathcal{U}\cdot\mathcal{V}$, $\mathcal{U}\otimes\mathcal{V}$ is an ultrafilter. Hence, since $\mathcal{U}\otimes\mathcal{V}\subseteq\mathcal{U}*\mathcal{V}$, $U\otimes\mathcal{V}=\mathcal{U}*\mathcal{V}$.

We extend the notation in a natural way: if $\mathcal{U}_i$ is an ultrafilter on A_i for $i\in I$, then $\bigotimes\{\mathcal{U}_i \mid i\in I\}$ is the filter consisting of all subsets of $\prod_{i\in I}A_i$ containing some set $\prod_{i\in I}U_i$, where $U_i\in\mathcal{U}_i$ for each $i\in I$.

An uncountable cardinal κ is *measurable* if there exists a κ-complete free ultrafilter on κ.

Lemma 4. *Let $\kappa_0,\kappa_1,\dots,\kappa_{n-1}$ be measurable cardinals with $\kappa_i<\kappa_{i+1}$ for $i<n-1$. Suppose that $\mathcal{U}_i$ is a κ_i-complete free ultrafilter on a set A_i, where $|A_i|<\kappa_{i+1}$ for $i<n-1$. Then $\bigotimes\{\mathcal{U}_i \mid i<n\}$ is a κ_0-complete free ultrafilter on $\prod_{i<n}A_i$.*

Proof. Apply induction on n. The base step, when $n=1$, is trivial. If the result holds for $n-1\geqslant 0$, then $\bigotimes\{\mathcal{U}_i \mid i<n-1\}$ is a κ_0-complete free ultrafilter on $\prod_{i<n-1}A_i$, and $|\prod_{i<n-1}A_i|=|A_{n-2}|<\kappa_{n-1}$. Hence $\mathcal{U}_{n-1}$ is $|\prod_{i<n-1}A_i|^+$-complete and, by Lemma 3 and the remark preceding it, $\bigotimes\{\mathcal{U}_i \mid i<n\}$ is a κ_0-complete, free ultrafilter.

The existence of measurable cardinals is necessary for Lemma 3 to work (with free ultrafilters). Recall that the existence of any countably complete free ultrafilter on any set implies the existence of an uncountable measurable cardinal.

Proposition 5. *Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on sets A and B which are not countably complete. Then $\mathcal{U} \cdot \mathcal{V} \neq \mathcal{U} * \mathcal{V}$.*

Proof. See [2, Corollary 7.24(b)].

Although the above result implies that $\cdot$ is not "commutative", it is associative.

Lemma 6. *Let $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ be filters on sets A, B and C, respectively. Then $\mathcal{U} \cdot (\mathcal{V} \cdot \mathcal{W}) = (\mathcal{U} \cdot \mathcal{V}) \cdot \mathcal{W}$.*

Proof. Let $S \in \mathcal{U} \cdot (\mathcal{V} \cdot \mathcal{W})$. Then $U \in \mathcal{U}$, where

$$U = \bigl\{a \in A \mid \bigl\{\langle b, c\rangle \mid \langle a, b, c\rangle \in S\bigr\} \in \mathcal{V} \cdot \mathcal{W}\bigr\}.$$

For each $a \in U$, let $S^a = \{\langle b, c\rangle \mid \langle a, b, c\rangle \in S\}$. Then each such S^a is in $\mathcal{V} \cdot \mathcal{W}$, so $V_a \in \mathcal{V}$, where $V_a = \{b \in B \mid \{c \mid \langle b, c\rangle \in S^a\} \in \mathcal{W}\}$. Put $T = \bigcup_{a \in U}\{a\} \times V_a$. Then $T \in \mathcal{U} \cdot \mathcal{V}$. If $\langle a, b\rangle \in T$ then $b \in V_a$, so $\{c \mid \langle a, b, c\rangle \in S\} \in \mathcal{W}$. Hence

$$T \subseteq \bigl\{\langle a, b\rangle \mid \bigl\{c \mid \langle a, b, c\rangle \in S\bigr\} \in \mathcal{W}\bigr\}.$$

Thus the latter set is in $\mathcal{U} \cdot \mathcal{V}$, so $S \in (\mathcal{U} \cdot \mathcal{V}) \cdot \mathcal{W}$. Hence $\mathcal{U} \cdot (\mathcal{V} \cdot \mathcal{W}) \subseteq (\mathcal{U} \cdot \mathcal{V}) \cdot \mathcal{W}$. The converse is similar.

Again, we can extend the notation to products of more than two ultrafilters, provided the index set is totally ordered. If $\mathcal{U}_i$ is an ultrafilter on A_i for each $i \in \omega$ then $\bigodot\{\mathcal{U}_i \mid i \in n\}$ is defined recursively for $n \geqslant 1$ by

$$\bigodot\{\mathcal{U}_i \mid i \in n\} = \begin{cases} \mathcal{U}_0 & \text{if } n = 1, \\ \Bigl(\bigodot\{\mathcal{U}_i \mid i \in n-1\}\Bigr) \cdot \mathcal{U}_{n-1} & \text{if } n > 1. \end{cases}$$

4. Intervals between regular topologies

We are now ready to prove the theorem mentioned in the abstract.

Theorem 7. *Assume that there exist infinitely many measurable cardinals. Let L be a finite lattice. Then there exists a set X and T_3 topologies σ and τ on X such that $L \cong [\sigma, \tau]$ if and only if L is distributive.*

Proof. As remarked in the Introduction, if L is isomorphic to a finite interval between T_3 topologies then L must be distributive.

Conversely, suppose L is a finite distributive lattice. Let $P = J(L)$, and index P as $\{a_i \mid i \in n\}$ (where $i \neq j$ implies $a_i \neq a_j$). For each $i \in n$ choose a measurable cardinal κ_i such that if $j < i$ then $\kappa_j < \kappa_i$, and a κ_i-complete free ultrafilter $\mathcal{U}_i$ on κ_i. For each $a \in P$ let $l(a) = \{i \in n \mid a_i \leqslant a\}$, let $g(a) = \{i \in n \mid a < a_i\}$ and let $r(a) = \{i \in n \mid a_i \nleqslant a$ and $(\exists b \in P)\ (a < b$ and $a_i \leqslant b)\}$. Let $S_a = \prod_{i \in l(a)} \kappa_i$. Choose some $p \notin \bigcup_{a \in P} S_a$, and let $X = \{p\} \cup \bigcup_{a \in P} S_a$.

For $x \in S_a$ and $U \in \prod_{i \in r(a)} \mathcal{U}_i$, define $B(x, U)$ to be the set

$$\{x\} \cup \bigcup_{b \in g(a)} \bigl\{y \in S_b \mid y \restriction l(a) = x \text{ and } (\forall j \in l(b) \setminus l(a))(y(j) \in U(j))\bigr\}.$$

For $U \in \prod_{i \in n} \mathcal{U}_i$, define $B(p, U)$ to be the set

$$\{p\} \cup \bigl\{x \in X \setminus \{p\} \mid (\forall i \in \operatorname{dom}(x))(x(i) \in U(i))\bigr\}.$$

The sets $B(x, U)$ form a local basis system at x for a T_3 topology σ on X—indeed, this topology is T_1 and zero-dimensional. To see that the topology is zero-dimensional (that is, has a base of clopen sets), we shall show that $B(x, U)$ is closed. To this end pick some z in S_b but not in $B(x, U)$. Since the argument for $x = p$ is similar, we can assume that $x \neq p$. If $g(b)$ and $g(a)$ are disjoint, then, for any $V \in \prod_{i \in r(b)} \mathcal{U}_i$, $B(z, V)$ and $B(x, U)$ are disjoint. On the other hand suppose that there is some c in both $g(b)$ and $g(a)$. If b is in $g(a)$, then either $z \restriction l(a) \neq x$, so that $B(x, U)$ and $B(z, U \restriction r(b))$ are disjoint, or $z \restriction l(a) = x$ but, for some k in $l(b) \setminus l(a)$, $z(k)$ is not in $U(k)$. In this latter case if y is in $B(z, V)$ for some V, $y(k) = z(k)$ is not in $U(k)$, hence y is not in $B(x, U)$. If b is not in $g(a)$ and $a = a_i$, then i is in $r(b)$. Let $V \in \prod_{i \in r(b)} \mathcal{U}_i$ be such that $V_i = \kappa_i \setminus \{x(i)\}$ and $V_j = \kappa_j$ for $j \neq i$. Then $B(z, V)$ misses $B(x, U)$. This completes the proof that $B(x, U)$ is closed.

Since $\mathcal{N}_\sigma(x) \restriction S_b = \{x \cup z \mid z \in \bigotimes\{\mathcal{U}_i \mid i \in l(b) \setminus l(a)\}\}$, if $x \in S_a$ and $a < b$, the topology has the property that $\overline{S_b}^\sigma = \{p\} \cup \bigcup_{a \leqslant b} S_a$. Moreover, for any $a \in P$ we have

$$\mathcal{N}_\sigma^o(p) \restriction S_a = \mathcal{N}_\sigma(p) \restriction S_a = \bigotimes \bigl\{\mathcal{U}_i \mid i \in l(a)\bigr\},$$

which is an ultrafilter by Lemma 4. Hence, by Lemma 2, if we put $\tau = \langle \sigma, \{p\} \rangle$ then $[\sigma, \tau] \cong \mathcal{O}(P) = \mathcal{O}(J(L)) \cong L$, as required.

The hypothesis of the existence of infinitely many measurable cardinals seems very strong. In ZFC, we can realize $\mathcal{O}(P)$ as an interval between T_3 topologies provided P is a disjoint union of trees.

Definition 8. A *copse* is a finite partially ordered set which is a disjoint union of trees.

Theorem 9. *Let P be a copse. Then $\mathcal{O}(P)$ is realizable as an interval between* T_3 *topologies.*

Proof. Index P as $\{a_i \mid i \in n\}$ in such a way that if $a_i < a_j$ in P then $i < j$. For each $a \in P$ let $l(a) = \{i \in n \mid a_i \leqslant a\}$, let $g(a) = \{i \in n \mid a < a_i\}$ and let $c(a) = \{b \in P \mid b \text{ covers } a \text{ in } P\}$. Let $S_a = \prod_{i \in l(a)} \omega$. Choose some $p \notin \bigcup_{a \in P} S_a$. As before, we construct a T_2 topology σ on $X = \{p\} \cup \bigcup_{a \in P} S_a$ to satisfy the conditions of Lemma 2.

Let $\mathcal{U}$ be a free ultrafilter on ω. For $x \in S_a$ and $U \in \mathcal{U}$, let $C(x, U)$ be the set

$$\{x\} \cup \bigcup_{b \in c(a)} \bigl\{y \in S_b \mid y \restriction l(a) = x \text{ and } y(b) \in U\bigr\}.$$

Let P_0 be the set of minimal elements of P. For $U \in \mathcal{U}$, let $C(p,U)$ be the set

$$\{p\} \cup \bigcup_{a \in P_0} \{x \in S_a \mid x(a) \in U\}.$$

Observe that if $U, V \in \mathcal{U}$ then $C(x,U) \cap C(x,V) = C(x, U \cap V)$. Thus the sets $C(x,U)$ for $x \in X$ and $U \in \mathcal{U}$ form a weak neighbourhood system for some topology σ (in other words, a set W is open in σ if and only if for every $x \in W$ there is some $U \in \mathcal{U}$ with $C(x,U) \subseteq W$). Note that the sets $C(x,U)$ are not open in σ (unless $<$ is a trivial partial order).

For $x \in X$ and $U \in \mathcal{U}$, let $B(x,U)$ be the set

$$\{x\} \cup \{z \in X \mid (\exists y \in C(x,U) \setminus \{x\})(z \restriction \mathrm{dom}(y) = y)\}.$$

Notice that if $y \in B(x,U) \setminus \{x\}$ then $C(y,\omega) \subseteq B(x,U)$, and that $C(x,U) \subseteq B(x,U)$. Thus $B(x,U)$ is open in σ.

This topology is clearly T_1. For $W \in \sigma$, let $W^* = W \setminus \{x \in W \mid (\exists y \in W \setminus \{x\})\ (\exists U \in \mathcal{U})\ (x \in C(y,U))\}$. Notice that $\{W \in \sigma \mid W^* = \{x\}\}$ forms a local basis at x and that W is clopen if $|W^*| = 1$, by a similar argument to the proof of Theorem 7. Hence the topology is zero-dimensional and therefore T_3.

Clearly if $a \in P$ and $b \in c(a)$ then $S_a \subseteq \overline{S_b}^{\sigma}$. From this it follows that $S_a \subseteq \overline{S_b}^{\sigma}$ for every $a < b$. On the other hand, if $a \nleqslant b$ then $\bigcup_{x \in S_a} B(x,\omega)$ is an open set containing S_a and missing S_b. Since we also have $p \in \overline{S_a}^{\sigma}$ for every $a \in P_0$, and hence for every $a \in P$, we have $\overline{S_b}^{\sigma} = \{p\} \cup \bigcup_{a \leqslant b} S_a$.

Finally, we can easily show by induction on the level of a in P that $\mathcal{N}^o_\sigma(p) \restriction S_a = \mathcal{N}_\sigma(p) \restriction S_a = \bigodot\{\mathcal{U} \mid i \in l(a)\}$, which is an ultrafilter. Thus, putting $\tau = \langle \sigma, \{p\}\rangle$, Lemma 2 shows that $[\sigma,\tau] \cong \mathcal{O}(P)$, as required.

5. Why assume the existence of measurable cardinals?

In this section we show that the construction given in Lemma 2 is canonical for lattices of T_1 topologies and discuss the relevance of measurable cardinals to the problem of realizing distributive lattice with T_3 topologies.

An interval $[\sigma,\tau]$ in the lattice of topologies on X is *basic* [6] if $\sigma < \tau$ and there is some $p \in X$ such that $\mathcal{N}_\sigma(x) = \mathcal{N}_\tau(x)$ for every $x \in X \setminus \{p\}$. We call p the *base* of the interval $[\sigma,\tau]$.

Lemma 10.

(1) *Let σ be a T_1 topology on X and let τ be a topology which covers σ in the lattice of topologies. Then $[\sigma,\tau]$ is basic.*

(2) *For $i = 1,2$ let L_i be a finite lattice isomorphic to a basic interval $[\sigma_i,\tau_i]$ on the set X_i based at p_i. Suppose X_1 and X_2 are disjoint, and let Y be the quotient set obtained from $X_1 \cup X_2$ by identifying the points p_1 and p_2. For $\mu \in [\sigma_1,\tau_1]$ and $\nu \in [\sigma_2,\tau_2]$ let $\mu * \nu$ be the quotient topology derived from the topology*

$\{U \cup V \mid U \in \mu$ and $V \in \nu\}$ under this identification. Then $[\sigma_1 * \sigma_2, \tau_1 * \tau_2]$ is isomorphic to $L_1 \times L_2$.

(3) *Let L be a finite lattice which is isomorphic to some interval between T_1 (T_3) topologies on some set X. Then L is isomorphic to a basic interval between T_1 (respectively T_3) topologies on some set Y.*

Proof. For (1) suppose there are distinct points p and q such that $\mathcal{N}_\sigma(p) \neq \mathcal{N}_\tau(p)$ and $\mathcal{N}_\sigma(q) \neq \mathcal{N}_\tau(q)$. Let U be some τ-open neighbourhood of p which is not a σ-neighbourhood of p. Let $\mu = \langle \sigma, U \setminus \{q\}\rangle$. Then $\sigma \subseteq \mu \subseteq \tau$. Since U is a μ-neighbourhood of p, $\sigma \neq \mu$. On the other hand, since $q \notin U \setminus \{q\}$, $\mathcal{N}_\mu(q) = \mathcal{N}_\sigma(q)$. Thus $\mu \neq \tau$. This contradicts the assumption that τ covers σ.

The proof of (2) is straightforward.

To prove (3), suppose that L is isomorphic to the interval $[\sigma, \tau]$ of T_1 topologies on X. Notice that, since L is finite, if $p \in X$ with $\mathcal{N}_\sigma(p) \neq \mathcal{N}_\tau(p)$, then there must be some $\mu, \nu \in [\sigma, \tau]$ such that ν covers μ and $\mathcal{N}_\mu(p) \neq \mathcal{N}_\nu(p)$. By part (1), p is the only point at which μ and ν differ. So there are finitely many points, $p_1, p_2, \ldots, p_n$ say, such that for any $x \notin \{p_1, p_2, \ldots, p_n\}$, $\mathcal{N}_\sigma(x) = \mathcal{N}_\tau(x)$. For each $i \in \{1, 2, \ldots, n\}$ choose an open set U_i such that $U_i \cap \{p_1, p_2, \ldots, p_n\} = \{p_i\}$. For all $\mu \in [\sigma, \tau]$, let $\mu_i = \mu \restriction U_i$. Then one can easily show that $[\sigma, \tau]$ is isomorphic to the product $\prod_{i=1}^n [\sigma_i, \tau_i]$, via the isomorphism $\mu \mapsto \langle \mu_i \mid 1 \leqslant i \leqslant n\rangle$. Let Y be the result of taking the disjoint union of the sets U_i and then identifying the points p_i. By part (2), if $\hat{\sigma}$ and $\hat{\tau}$ are the topologies on Y induced in the natural way by σ and τ, respectively, then $[\hat{\sigma}, \hat{\tau}]$ is also isomorphic to $\prod_{i=1}^n [\sigma_i, \tau_i]$, and therefore to the original lattice L. Notice that if σ and τ are T_3, then so are $\hat{\sigma}$ and $\hat{\tau}$.

We observe that the above lemma, together with Valent and Larson's result that a finite lattice is isomorphic to a basic interval between T_1 topologies if and only if it is distributive, together imply Rosický's result that every finite interval between T_1 topologies is distributive.

Lemma 11. *Let Φ be an isomorphism from the finite distributive lattice L to the basic interval of T_1 topologies $[\sigma, \tau]$ on the set X. Then for each a and b in $J(L)$ there are sets U_a such that*

(1) *$\Phi(a) = \langle \sigma, U_a\rangle$, and*
(2) *$U_a \subseteq U_b$ if and only if $b \leqslant a$.*

Proof. Let p be the base of $[\sigma, \tau]$. For each $a \in J(L)$, let b_a be the unique element covered by a, let $\mu_a = \Phi(a)$, and let $\nu_a = \Phi(b_a)$. Choose some $V_a \in \mu_a \setminus \nu_a$. Then $\nu_a < \langle \nu_a, V_a\rangle \leqslant \mu_a$, and μ_a covers ν_a, so $\langle \nu_a, V_a\rangle = \mu_a$.

For each $a \in J(L)$, let $U_a = \bigcap\{V_b \mid b \in J(L)$ and $b \leqslant a\}$. Notice that if $b \leqslant a$ then $V_b \in \mu_b \leqslant \mu_a$, so $U_a \in \mu_a$. Thus $\mu_a \geqslant \langle \sigma, U_a\rangle$. We prove the converse by induction on $n_a = |\{b \in J(L) \mid b < a\}|$. If $n_a = 0$, then $\nu_a = \sigma$, and the result is trivial. So suppose $n_a > 0$ and the result holds for all b with $n_b < n_a$. In particular, it holds for all $b \in J(L)$ with $b < a$. Let $W \in \mu_a$. If $p \notin W$ then $W \in \sigma$. So suppose $p \in W$. Then there is some

$U \in \nu_a$ with $p \in U \cap V_a \subseteq W$. Now $U \in \nu_a = \bigvee\{\mu_b \mid b \in J(L)$ and $b < a\}$, so by inductive hypothesis there is some $U' \in \sigma$ with $p \in U' \cap \bigcap\{U_b \mid b \in J(L)$ and $b < a\} \subseteq U$. Then $p \in U' \cap U_a \subseteq W$, so $W \in \langle \sigma, U_a \rangle$ as required.

Finally, we must show that for $a, b \in J(L)$, $U_a \subseteq U_b$ if and only if $b \leqslant a$. From the construction it is clear that if $b \leqslant a$ then $U_a \subseteq U_b$. Conversely, if $U_a \subseteq U_b$ then $p \in U_a \subseteq U_b$, so U_b is a neighbourhood of p in $\langle \sigma, U_a \rangle$. Thus $U_b \in \langle \sigma, U_a \rangle$, so $\langle \sigma, U_b \rangle \subseteq \langle \sigma, U_a \rangle$, i.e., $\mu_b \leqslant \mu_a$, so $b \leqslant a$.

Lemma 12. *Let μ and ν be T_1 topologies on a set X such that μ covers ν. If $S \subseteq X$ and $p \in \overline{S}^{\nu} \setminus \overline{S}^{\mu}$ then $\mathcal{N}^o_\nu(p) \restriction S$ is an o-ultrafilter on S.*

Proof. Suppose $\mathcal{N}^o_\nu(p) \restriction S$ is not an o-ultrafilter. Let $\mathcal{F}$ be a strictly finer o-filter on S, and choose some $U \in \mathcal{F} \setminus (\mathcal{N}^o_\nu(p) \restriction S)$. Then there is some $V \in \mu$ with $V \cap S = U$. Put $W = V \cup (X \setminus \overline{S}^{\mu})$. Then $p \in W \in \mu$. Put $\theta = \langle \nu, W \rangle$. Then $\nu \leqslant \theta \leqslant \mu$. Since $U = W \cap S \in \mathcal{N}^o_\theta(p) \restriction S$, $\theta \neq \nu$. On the other hand, $\mathcal{N}^o_\theta(p) \restriction S \subseteq \mathcal{F}$, and $\mathcal{F}$ is a proper o-filter on S, so $p \in \overline{S}^{\theta}$. Thus $\theta \neq \mu$. So $\nu < \theta < \mu$, contradicting the assumption that μ covers ν.

Theorem 13. *Let L be a finite lattice which is realized as a basic interval between T_1 topologies σ and τ on a set Y. Then there is a subset X of Y such that $(X, \sigma \restriction X)$ has the form described in Lemma 2 (with $P = J(L)$).*

Proof. Let $P = J(L)$. Suppose that σ and τ are T_1 topologies on Y such that $[\sigma, \tau]$ is a basic interval based at p, and that $\Phi : L \to [\sigma, \tau]$ is an isomorphism. By Lemma 11, we can find sets U_a for $a \in P$ such that $\Phi(a) = \langle \sigma, U_a \rangle$ for every $a \in P$, and $U_a \subseteq U_b$ whenever $b \leqslant a$.

For each $a \in P$ let $\mu_a = \Phi(a)$, let $n(a) = \{b \in P \mid a \nleqslant b\}$, and let $R_a = (X \setminus U_a) \cap \bigcap_{b \in n(a)} U_b$. Observe that if $b \in n(a)$ then $R_a \subseteq U_b$ and $R_b \subseteq X \setminus U_b$, so R_a and R_b are disjoint. Since we have $a \in n(b)$ or $b \in n(a)$ for every $a, b \in P$ with $a \neq b$, the sets R_a are disjoint.

Claim. *If $a < b$ then there is some $V \in \sigma$ such that $p \in V$ and $V \cap R_a \subseteq \overline{R_b}^{\sigma}$.*

Proof. For put $W = (X \setminus \overline{R_b}^{\sigma}) \cup \bigcap_{c \in P} U_c$, and put $\theta = \langle \sigma, W \rangle$. If there is no such V then, for every $V \in \sigma$ with $p \in V$, $V \cap R_a \cap W \neq \emptyset$. Thus, there is no $V \in \sigma$ with $p \in V \cap W \subseteq U_a$, so $U_a \notin \theta$ and $\mu_a \nless \theta$.

On the other hand, since $W \cap R_b = \emptyset$, $W \cap (X \setminus U_b) \cap \bigcap_{c \in n(b)} U_c = \emptyset$. Therefore p is in $W' = W \cap \bigcap_{c \in n(b)} U_c \subseteq U_b$ and $U_b \in \langle \sigma, W' \rangle$. Hence $\langle \sigma, U_b \rangle \leqslant \langle \sigma, W' \rangle$, in other words, $\mu_b \leqslant \theta \vee \bigvee_{c \in n(b)} \mu_c$. Since μ_b is join-irreducible, this implies that $\mu_b \leqslant \theta$ or $\mu_b \leqslant \mu_c$ for some $c \in n(b)$. The latter would contradict the assumption that Φ is an order-isomorphism, so $\mu_b \leqslant \theta$.

Thus we have $\mu_a \leqslant \mu_b \leqslant \theta$, but $\mu_a \nleqslant \theta$, a contradiction.

For each $a, b \in P$ with $a < b$, choose some $V_{a,b} \in \sigma$ such that $V_{a,b} \cap R_a \subseteq \overline{R_b}^{\sigma}$. Let $V_{a,a} = Y$ and $V = \bigcap\{V_{a,b} \mid a, b \in P$ and $a \leqslant b\}$. Then $V \in \sigma$, so if $x \in V \cap \overline{A}^{\sigma}$ then $x \in \overline{V \cap A}^{\sigma}$. In particular, if $a < b$ then, since $V \subseteq V_{a,b}$, $V \cap R_a \subseteq \overline{V \cap R_b}^{\sigma}$.

Put $S_a = V \cap R_a$ for each $a \in P$. Then, by the above comment, if $a \leqslant b$ then $S_a \subseteq \overline{S_b}^{\sigma}$. On the other hand, if $a \nleqslant b$ then $b \in n(a)$, so

$$S_a \subseteq R_a \subseteq U_b \setminus \{p\} \subseteq X \setminus R_a \subseteq X \setminus S_a,$$

and thus $S_a \cap \overline{S_b}^{\sigma} = \emptyset$.

Claim. *For every $a \in P$, $p \in \overline{S_a}^{\sigma}$.*

Proof. For suppose not. Choose U such that $p \in U \in \sigma$ and $U \cap S_a = \emptyset$. Then $U \cap V \cap R_a = \emptyset$, in other words $U \cap V \cap (X \setminus U_a) \cap \bigcap_{b \in n(a)} U_b = \emptyset$. But then $p \in (U \cap V) \cap \bigcap_{b \in n(a)} U_b \subseteq U_a$, so $U_a \in \langle \sigma, \bigcap_{b \in n(a)} U_b \rangle \leqslant \bigvee_{b \in n(a)} \langle \sigma, U_b \rangle$ and $\mu_a \leqslant \bigvee_{b \in n(a)} \mu_b$. Since μ_a is join-irreducible, $\mu_a \leqslant \mu_b$ for some $b \in n(a)$. As before, this contradicts the assumption that Φ is an order-isomorphism.

Now, if we put $X = \{p\} \cup \bigcup_{a \in P} S_a$, we have $\overline{S_b}^{\sigma \restriction X} = \{p\} \cup \bigcup\{S_a \mid a \in P$ and $a \leqslant b\}$, as required. To finish, we only need to show that $\mathcal{N}^{o}_{\sigma \restriction X}(p) \restriction S_a$ is an o-ultrafilter for every $a \in P$. By Lemma 12, it is enough to show that $p \in \overline{S_a}^{\nu_a} \setminus \overline{S_a}^{\mu_a}$, where ν_a is the unique topology covered by μ_a in $[\sigma, \tau]$. Since $p \in U_a$ and $U_a \cap S_a = \emptyset$, we have $p \notin \overline{S_a}^{\mu_a}$. On the other hand, if $p \notin \overline{S_a}^{\nu_a}$, then we can find some $U \in \nu_a$ with $p \in U$ and $U \cap S_a = \emptyset$. Since $\nu = \bigvee\{\mu_b \mid b \in P$ and $b < a\}$, this means that we can find some $W \in \sigma$ with $p \in W \cap \bigcap_{b \in n(a)} U_b \subseteq Y \setminus S_a$, and as before this implies that $\mu_a \leqslant \bigvee_{b \in n(a)} \mu_b$, a contradiction.

Thus this subset X has the properties claimed.

To explain the relevance of measurable cardinals we shall discuss a specific example: Let Λ denote the three-element partially ordered set $\{a, b, c\}$ with $a < c$, $b < c$ and no other non-trivial relations. Clearly $L = \mathcal{O}(\Lambda)$ is the smallest distributive lattice for which $J(L)$ is not a copse.

Suppose we can realize $\mathcal{O}(\Lambda)$ as an interval between T_3 topologies. By Theorem 13, we can assume that the realization is of the form described in Lemma 2. So we have a point p, and three disjoint sets S_a, S_b and S_c. We also have three o-ultrafilters $\mathcal{U}_a, \mathcal{U}_b$ and $\mathcal{U}_c$, on S_a, S_b and S_c, respectively. If we assume that S_a, S_b and S_c are relatively discrete, then these o-ultrafilters are in fact ultrafilters. These ultrafilters have the property that whenever an open set contains $\mathcal{U}_a$ many points of S_a, it must also contain $\mathcal{U}_c$ many points of S_c. Similarly, whenever an open set contains $\mathcal{U}_b$ many points of S_b it must also contain $\mathcal{U}_c$ many points of S_c. We also know that points of S_a and points of S_b have disjoint neighbourhoods, since the topology is T_2. The natural way to arrange this is to associate S_c with $S_a \times S_b$, and $\mathcal{U}_c$ with the product of $\mathcal{U}_a$ and $\mathcal{U}_b$. In this case $\mathcal{U}_c$ is $\mathcal{U}_a \cdot \mathcal{U}_b$ when S_c is thought of in one way and $\mathcal{U}_a * \mathcal{U}_b$ when thought of it in the other. As indicated in Section 3, these two are distinct ultrafilters unless there exists a measurable cardinal.

Of course, by the result in [3], $\mathcal{O}(\Lambda)$ is realizable as an interval between T_2 topologies without the assumption of measurable cardinals. However, the approach used in that paper does not yield regular topologies.

6. Not all the topologies in an interval between T_3 topologies are T_3

Lemma 14. *Let L_3 denote the 3-element linear order, and let σ, τ be T_2 topologies on some set X such that $[\sigma, \tau] \cong L_3$. Let μ be the unique topology with $\sigma < \mu < \tau$. Then μ is not T_3.*

Proof. By Lemma 10, we know that both $[\sigma, \mu]$ and $[\mu, \tau]$ are basic. Let p and q be the bases of these two intervals. If $p \neq q$ then, by the same lemma, $[\sigma, \tau]$ would be isomorphic to the product $[\sigma, \mu] \times [\mu, \tau]$, which it is not. So we must have $p = q$. Let $U, V \in \tau$ with $p \in U \subseteq V$ and $\mu = \langle \sigma, V \rangle$, $\tau = \langle \sigma, U \rangle$. Let $A = V \setminus U$, and let $B = X \setminus V$. Suppose μ is T_3. Then there is some $W \in \mu$ with $p \in W \subseteq \overline{W}^{\mu} \subseteq V$. Notice that, since $p \in W$, $\overline{W}^{\sigma} = \overline{W}^{\tau} = \overline{W}^{\mu}$. Thus, putting $T = X \setminus \overline{W}^{\mu}$, we have $B \subseteq T$ and $T \in \sigma$.

Let $\nu = \langle \sigma, W \rangle$ and let $\theta = \langle \sigma, U \cup T \rangle$. Since $W \in \mu$, $\sigma \leqslant \nu \leqslant \mu$. On the other hand, since $p \in W \subseteq V$, $\langle \sigma, V \rangle \leqslant \langle \sigma, W \rangle$. Thus $\mu \leqslant \nu$, so $\mu = \nu$. Now, if $U \cup T \in \mu$ then $(U \cup T) \cap W = U \in \mu$, contradicting the assumption that $\mu < \tau = \langle \sigma, U \rangle$. So $U \cup T \notin \mu$, and therefore $\theta \nleqslant \mu$. Thus we must have $\theta = \tau$. In particular, we have $U \in \theta$, so there is some $S \in \sigma$ with $p \in S \cap (U \cup T) \subseteq U$. But then $S \cap B = \emptyset$ and $p \in S \subseteq V$, so $V \in \sigma$, contradicting the assumption that $\sigma < \mu = \langle \sigma, V \rangle$.

Proposition 15. *Let P be a non-trivial finite partial order. If $[\sigma, \tau]$ is an interval in the lattice of topologies on some set, and $[\sigma, \tau]$ is isomorphic to $\mathcal{O}(P)$, then $[\sigma, \tau]$ contains a topology which is not T_3.*

Proof. Suppose $\mathcal{O}(P)$ is isomorphic to the interval $[\sigma, \tau]$, via the isomorphism φ.

Since P is non-trivial, it contains some elements a and b with $a < b$. Without loss of generality we may assume that b covers a. Put $T = \{x \in P \mid x \leqslant b\}$, and $S = T \setminus \{a, b\}$. Then the interval $[S, T]$ in $\mathcal{O}(P)$ is isomorphic to L_3, so by Lemma 14 either $\varphi(S)$ is not T_2 or $\varphi(S \cup \{a\})$ is not T_3.

7. Restricting to countable sets

Finally we consider the effect of restricting the cardinality of the underlying set. In particular, what finite lattices can be realized as intervals between topologies on a countable set?

We will show that the restriction to countable sets does not affect the situation for intervals between T_1 topologies: any finite distributive lattice is isomorphic to an interval between T_1 topologies on a countable set. On the other hand, a finite distributive lattice L

can be realized as an interval between T_3 topologies on a countable set if and only if $J(L)$ is a copse.

Theorem 16. *Let L be a finite, distributive lattice. Then there exist T_1 topologies σ and τ on a countable set X such that $[\sigma, \tau]$ is isomorphic to L.*

Proof. Let $P = J(L)$. For $a \in P$, let $c(a) = \{b \in P \mid b$ covers a in $P\}$. Define a function $l : P \to \omega$ by

$$l(a) = \begin{cases} 1 & \text{if } a \text{ is minimal in } P, \\ \max\{l(b) + 1 \mid b \in P \text{ and } a \in c(b)\} & \text{otherwise.} \end{cases}$$

For $a \in P$ let $S_a = \{a\} \times \omega^{l(a)}$. Choose some $p \notin \bigcup_{a \in P} S_a$. Let $X = \{p\} \cup \bigcup_{a \in P} S_a$. As in the proof of Theorem 9, we will specify a topology σ on X by describing the weak neighbourhoods of points of X.

Let $\mathcal{U}$ be a free ultrafilter on ω. A weak neighbourhood of p consists of p together with all points $\langle a, n\rangle$ such that a is minimal in P and $n \in U$, for some $U \in \mathcal{U}$. A weak neighbourhood of $x = \langle a, f\rangle$ consists of x together with a subset of S_b for each $b \in c(a)$: for each $b \in c(a)$ we choose some $U \in \bigodot\{\mathcal{U} \mid i \in l(b) \setminus l(a)\}$, and include all the points $\langle b, f \cup g\rangle$ for $g \in U$.

This topology is T_1 but is not T_2 (unless P is a copse, in which case it is the same as that constructed in the proof of Theorem 9). By Lemma 6, $\mathcal{N}^o_\sigma(p) \restriction S_a = \mathcal{N}_\sigma(p) \restriction S_a = \bigodot\{\mathcal{U} \mid i \in l(a)\}$. Since we clearly have $\overline{S_a}^\sigma = \{p\} \cup \bigcup_{b \leqslant a} S_b$, by Lemma 2 we have $[\sigma, \langle\sigma, \{p\}\rangle] \cong \mathcal{O}(P) \cong L$.

Theorem 17. *Let L be a finite lattice. Then L is isomorphic to an interval between two T_3 topologies on a countable set if and only if $L \cong \mathcal{O}(P)$ for some copse P.*

Proof. If $L \cong \mathcal{O}(P)$ for some copse P then, by the construction given in the proof of Theorem 9, L is realizable as such an interval.

Conversely, suppose that L is not isomorphic to $\mathcal{O}(P)$ for any copse P. Either L is not distributive, in which case it is not even isomorphic to an interval between T_1 topologies, or $J(L)$ is not a copse. So assume the latter holds. Then there exist $a, b, c \in J(L)$ with $a \nleqslant b$, $b \nleqslant a$, $a < c$ and $b < c$.

Suppose that L is indeed realizable as an interval between T_3 topologies σ and τ on a countable set X, via an isomorphism φ. By Lemma 10, we can assume that $[\sigma, \tau]$ is a basic interval, based at p. By Theorem 13, we can find a subspace of X which has the form given in Lemma 2. Now $S_a \cap \overline{S_b}^\sigma = \emptyset = \overline{S_a}^\sigma \cap S_b$. Since σ is a T_3 topology on a countable set, it is hereditarily normal, so we can find disjoint σ-open sets W_a and W_b containing S_a and S_b, respectively. By Lemma 11 we can find sets U_a, U_b and U_c such that $\varphi(a) = \langle\sigma, U_a\rangle$, $\varphi(b) = \langle\sigma, U_b\rangle$, $\varphi(c) = \langle\sigma, U_c\rangle$ and $U_c \subseteq U_a \cap U_b$ (and $S_a \cap U_a = \emptyset$ etc.).

Let $\mu = \langle\sigma, U_c \cup W_b\rangle$ and let $\nu = \langle\sigma, U_c \cup W_a\rangle$. Then $\mu \leqslant \varphi(a)$ and $\nu \leqslant \varphi(b)$, so $\mu \vee \nu \leqslant \varphi(a \vee b) < \varphi(c)$. On the other hand, $U_c = (U_c \cup W_b) \cap (U_c \cup W_a) \in \mu \vee \nu$, so $\varphi(c) = \langle\sigma, U_c\rangle \leqslant \mu \vee \nu$, a contradiction.

The problem of characterising the finite lattices which can be realized as an interval between T_2 topologies on a countable set is still open. Our conjecture is that $\mathcal{O}(\Lambda)$ is not realizable as such an interval, in which case a similar argument to the above should yield a similar characterisation to that for intervals between T_3 topologies on a countable set.

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